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Harmonic Interpolation and Lie Groups

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A harmonic interpolation of a polygon (for odd and even numbers of points forming the polygon) used in computer graphics is derived from the primary permutation matrix using the spectral decomposition of the matrix. This matrix can be used to generate an orthonormal basis in the Hilbert space of all $n \times n$ matrices over $\mathbf{C}$. The connection with Lie groups is discussed.

KEY WORDS: Lie groups; harmonic interpolation; permutation matrices.

1. INTRODUCTION

In computer graphics and interpolation, one of the most important tasks is to find a smooth curve (harmonic interpolation) through a number of n-ordered points (a polygon) $\mathbf{X} = (X_0, X_1, \ldots, X_{n-1})^T$ given in the Hilbert space $\mathbf{R}^d$, i.e., $X_j \in \mathbf{R}^d$. In the construction the starting point is the $n \times n$ primary permutation matrix and its spectral decomposition. In this paper we describe the construction of the harmonic interpolation starting from the primary permutation matrix and the spectral representation of this matrix. We are not only going to study the case with an odd number of points forming a polygon (Schuster, 2001), but also the case with an even number of points. furthermore we show that this matrix can be used to generate an orthonormal basis in the Hilbert space of all $n \times n$ matrices over $\mathbf{C}$ with the scalar product $\langle A|B\rangle := \mathrm{tr}(AB^*)/n$. The connection with Lie groups is also discussed. The primary permutation matrix is also the starting point in the construction of maximally entangled quantum states in finite dimensional Hilbert spaces (Ban, 2002; Hardy and Steeb, 2001; Hardy *et al.*, 2003; Steeb and Hardy, 2004). Another closely related problem is to consider a system composed by n identical masses placed at the vertices of a regular n-gon and the corresponding gravitational field (Bang and Elmabsout, 2003).

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2. PRIMARY PERMUTATION MATRIX AND PROPERTIES

The $n \times n$ primary permutation matrix U is given by

$$U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{1}$$

It is an $n \times n$ circulant matrix. An arbitrary $n \times n$ circulant matrix C (Steeb, 1997) is given by

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}. \tag{2}$$

Let us first summarize the properties of the matrix U. The set of matrices $\{U, U^2, \dots, U^n = I_n\}$ is a subgroup of the group of all $n \times n$ permutation matrices under matrix multiplication. They form a cyclic group of order n. An $n \times n$ matrix D is circulant if and only if $D = UDU^{\mathrm{T}}$, where U is given by (1). Let C be a circulant matrix given by (2) and let

$$f(\lambda) = c_0 + c_1\lambda + \dots + c_{n-1}\lambda^{n-1}. \tag{3}$$

Then $C = f(U)$, where U is the primary permutation matrix. The matrix C is normal, that is $C^*C = CC^*$. The eigenvalues of C are $f(\omega^k)$ where $k = 0, 1, \dots, n-1$ and $\omega := \exp(2\pi i/n)$. For the determinant we have $\det(C) = f(\omega^0)f(\omega^1)\dots f(\omega^{n-1})$. The matrix F^*CF is a diagonal matrix, where F is the unitary matrix with the (j, k)-entry equal to $\omega^{(j-1)(k-1)}$, for $j, k = 1, \dots, n$.

Consider the $n \times n$ unitary diagonal matrix

$$V = \mathrm{diag}(1, \omega, \omega^2, \dots, \omega^{n-1}) \tag{4}$$

where $\omega = \exp(2\pi i/n)$ and thus $\omega^n = 1$ Then the set $\{U^jV^k : j, k = 0, 1, \dots, n-1\}$ forms an orthonormal basis in the Hilbert space of all $n \times n$ matrices over $\mathbf{C}$ with the scalar product $\langle A|B\rangle := \mathrm{tr}(AB^*)/n$. Obviously, the matrices U^jV^k are unitary, since U and V are unitary. For $n = 2$ we find the orthonormal basis $\{I_2, \sigma_x, \sigma_z, i\sigma_y\}$, where $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli spin matrices. We also have $UV = \omega VU$. This relation plays an important role in quantum theory in finite dimensional Hilbert spaces.

3. SPECTRAL DECOMPOSITION

For our construction we need the eigenvalues and normalized eigenvectors of U. The eigenvalues are given by $\lambda^0 = 1, \lambda^1, \lambda^2, \ldots, \lambda^{n-1}$ with

$$\lambda := \exp(2\pi i/n) \tag{5}$$

The normalized eigenvectors are given by

$$|\theta_j\rangle = \frac{1}{\sqrt{n}}(1, \exp(-i2\pi j/n), \exp(-i4\pi j/n), \ldots, \exp(-i2\pi(n-1)j/n))^{\mathrm{T}} \tag{6}$$

where $j = 0, 1, \ldots, n-1$. Thus for the eigenvalue 1 of U we find the normalized eigenvector

$$|\theta_0\rangle = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^{\mathrm{T}}. \tag{7}$$

The eigenstates $\{|\theta_j\rangle : j = 0, 1, \ldots, n-1\}$ and the standard basis $\{|\mathbf{e}_j\rangle\}(j = 0, 1, \ldots, n-1)$ in $\mathbf{C}^n$ are connected by the discrete Fourier transform

$$|\theta_j\rangle = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\exp(-ik\theta_j)|\mathbf{e}_k\rangle, \quad |\mathbf{e}_k\rangle = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\exp(ik\theta_j)|\theta_j\rangle \tag{8}$$

where $\theta_j := 2\pi j/n$. Thus the spectral representation of U can be written as

$$U = \sum_{j=0}^{n-1}\lambda^j P_j \equiv \sum_{j=0}^{n-1}\lambda^j|\theta_j\rangle\langle\theta_j|. \tag{9}$$

The projection matrix $P_j = |\theta_j\rangle\langle\theta_j|$ can be expressed using U^k. Since $P_j P_k = 0$ for $j \neq k$ and $P_j^2 = P_j$ we find

$$U^k = \sum_{j=0}^{n-1}\lambda^{jk}P_j, \quad k = 1, 2, \ldots, n. \tag{10}$$

Thus we calculated the Fourier transform of the projection matrices. For $k = n$ we have $U^n = I_n$ and the completeness relation

$$I_n = \sum_{j=0}^{n-1}P_j. \tag{11}$$

The inverse of the matrix (λ^{jk}) is given by $\frac{1}{n}(\lambda^{-jk})$. Thus

$$P_j = \frac{1}{n}\sum_{k=0}^{n-1}\lambda^{-jk}U^k, \quad j = 0, 1, \ldots, n-1. \tag{12}$$

4. HARMONIC INTERPOLATION

For the construction of the harmonic interpolation in computer graphics we proceed as follows. We embed the matrices U^k into a real Lie group (Steeb, 1996). Thus we have to consider the cases $n = 2m + 1$ and $n = 2m$ where m is a positive integer. In the second case we have $\lambda^0 = 1$ and $\lambda^m = -1$. Furthermore we define $P_{-j} := P_{n-j}$. The projection matrices are therefore real.

We consider first the case $n = 2m + 1$ (Schuster, 2000). We obtain by replacing k by t

$$U(t) = P_0 + \sum_{j=1}^{m}(\lambda^{jt} P_j + \lambda^{-jt} P_{-j}) \equiv P_0 + \sum_{j=1}^{m}(e^{2\pi ijt/n} P_j + e^{-2\pi ijt/n} P_{-j}). \tag{13}$$

The unitary matrices $U(t)$ are n-periodic, i.e., $U(t + n) = U(t)$. We replace t by nt and therefore we reduce the period to 1. Thus $U(t)$ takes the form

$$\tilde{U}(t) = P_0 + \sum_{j=1}^{m}(e^{2\pi ijt} P_j + e^{-2\pi ijt} P_{-j}), \quad t \in \mathbf{R}. \tag{14}$$

Thus $\tilde{U}(t + 1) = \tilde{U}(t)$ and $\tilde{U}(k/n) = U^k$ for $k = 0, 1, \ldots, n - 1$. The $n \times n$ matrices $\tilde{U}(t)$ form a one-dimensional abelian unitary group (Lie group) with $\tilde{U}(0) = I_n$ and $\tilde{U}(t + \tau) = \tilde{U}(t)\tilde{U}(\tau)$. Owing to (12) we can write $\tilde{U}(t)$ as

$$\tilde{U}(t) = \frac{1}{n}\sum_{k=0}^{n-1}\left(1 + \sum_{j=1}^{m}\left(e^{2\pi ij(t-k/n)} + e^{-2\pi ij(t-k/n)}\right)\right)U^k. \tag{15}$$

We define

$$\sigma_k(t) \equiv \sigma(t - k/n) := \frac{1}{n}\left(1 + \sum_{j=1}^{m}\left(e^{2\pi ij(t-k/n)} + e^{-2\pi ij(t-k/n)}\right)\right). \tag{16}$$

Using $\exp(i\alpha) \equiv \cos(\alpha) + i\sin(\alpha)$,

$$1 + 2\sum_{j=1}^{m}\cos(j\alpha) \equiv \frac{\sin((m + 1/2)\alpha)}{\sin(\alpha/2)} \tag{17}$$

and with $n = 2m + 1$ we find

$$\sigma_k(t) = \frac{\sin(\pi n(t - k/n))}{n\sin(\pi(t - k/n))}, \quad k = 0, 1, \ldots, n - 1. \tag{18}$$

Equation (17) was used by Lejeune-Dirichlet for his proof of Fourier's formula (Lejeune-Dirichlet, 1829). Thus we can write

$$\tilde{U}(t) = \sum_{k=0}^{n-1}\sigma_k(t)U^k. \tag{19}$$

We have the properties

$$\sum_{k=0}^{n-1} \sigma_k(t) = 1, \quad \sum_{k=0}^{n-1} \sigma_k^2(t) = 1. \tag{20}$$

Thus the functions $\sigma_k(t)$ and $\sigma_k^2(t)$ provide a partition of unity. Now let $\mathbf{X} = (X_0, X_1, \ldots, X_{n-1})^{\mathrm{T}}$ be the vector which describes the polygon. Then

$$\mathbf{X}(t) = \tilde{U}(t)\mathbf{X} = \sum_{k=0}^{n-1} \sigma(t - k/n)U^k \mathbf{X}. \tag{21}$$

The curve which describes our closed smooth curve is given by

$$\begin{aligned} X_\ell(t) &= \sum_{k=0}^{n-1} \sigma(t - k/n)X_{k+\ell} = \sum_{k=0}^{n-1} \sigma(t + \ell/n - (k+\ell)/n)X_{k+\ell} \\ &= X_0(t + \ell/n) \end{aligned} \tag{22}$$

where $(k + \ell)$ is calculated mod n. Hence for all ℓ it represents the same curve and we can write

$$X(t) = \sum_{k=0}^{n-1} \sigma(t - k/n)X_k, \quad t \in [0, 1]. \tag{23}$$

Thus this curve (23) interpolates the points of the given polygon smoothly. We consider now the case $n = 2m$. For this case we can write

$$U^k = P_0 + (-1)^k P_m + \sum_{j=1}^{m-1} (\lambda^{jk} P_j + \lambda^{-jk} P_{-j}). \tag{24}$$

If we replace the discrete variable k by the real variable t, we find the factor $(-1)^t \equiv e^{\pi i t}$. The other terms are real. Thus the Lie group we would find is not real and therefore cannot be used directly for the harmonic interpolation in computer graphics. Since the function $U(t)$ will be continuous in the complex domain and go through the points of the polygon, the real part will also be continuous and go through the points of the polygon. A similar calculation as described for the case $n = 2m + 1$ given above yields ($t \in [0, 1]$)

$$\begin{aligned} \tilde{U}(t) = \frac{1}{n} \sum_{k=0}^{n=1} \cos(\pi(nt - k))U^k \\ + \frac{i}{n} \sum_{k=0}^{n-1} \sin(\pi(nt - k))U^k + \sum_{k=0}^{n-1} \xi_k(t)U^k \end{aligned} \tag{25}$$

where

$$\xi(t) := \frac{\sin(\pi t(n-1))}{n \sin(\pi t)}, \quad \xi_k(t) := \xi\left(t - \frac{k}{n}\right). \tag{26}$$

Thus our smooth curve in the even case which goes through all the points of the polygon is

$$X(t) = \sum_{k=0}^{n-1} \left(\frac{1}{n} \cos(\pi(nt - k)) + \xi_k(t) \right) X_k. \tag{27}$$

The curves described above are stable in the sense that making a small change in the position of a point will only cause a small change of the curve in the neighbourhood of this point. For an example of harmonic interpolation, we illustrate how a unit circle centered at the origin can be parameterized using harmonic interpolation. We use four control points: $\mathbf{X}_0 = (1, 0)$, $\mathbf{X}_1 = (0, 1)$, $\mathbf{X}_2 = (-1, 0)$ and $\mathbf{X}_3 = (0, -1)$. Then

$$\mathbf{p}(t) = \sum_{k=0}^{3} \left(\frac{1}{n} \cos(\pi(nt - k)) + \xi_k(t) \right) \mathbf{X}_k. \tag{28}$$

After simplification we get

$$\mathbf{p}(t) = \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}. \tag{29}$$

The usefulness of harmonic interpolation is demonstrated when compared to Bézier curves. It has been shown that Bézier curves cannot be used to draw a circle (Farin, 1996). A circle can however be described by three rational Bézier curves. A Java application and Java applet which draw these curves is available from the authors.

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